Problem 8

Define differential scattering cross section Show that

$$d\sigma = \frac{|M|^2}{F}\, dQ\,,$$

where M is the invariant amplitude for the process, dQ is Lorentz invariant phase space factor and F is incident flux factor.

(Consider two particle scattering: $A + B \rightarrow C + D$.)

The differential cross section is defined as

$$d\sigma = \frac{(\text{transition rate per unit volume}) \times (\text{number of final states})}{(\text{initial flux})}\,,$$

where

$$\text{transition rate per unit volume} \equiv W_{fi} = \frac{|T_{fi}|^2}{TV}$$

(the limits of time inteval for interaction $T \rightarrow \infty$, and volume $V \rightarrow \infty$ are understood).

For the scattering process $A + B \rightarrow C + D$, the transition amplitude is

$$T_{fi} = (-i\,M)\,(N_A\, N_B\, N_C\, N_D)\,(2\pi)^4\, \delta^4(p_D + p_C - p_A - p_B)$$

where the normalization constants are taken out of the invariant amplitude M.

Normalization choice

For fermions as well as bosons, we choose the covariant normalization in which we have 2E particles in volume V.

For scalar particle,

$\phi(x) = N\,e^{-i\,p\,.\,x}$, $\rho = 2\,E\,|\,N\,|^2\,V$ and $\int_V \rho\,dV = 2\,E\,|\,N\,|^2\,V$.

Requirement of 2E particles in volume V requires

$$N = \frac{1}{\sqrt{V}}\,.$$

For a fermion (spin ½)

$$\int_V \rho\,dV \;=\; \int_V \psi^\dagger\,\psi\,dV \;=\; u^\dagger\,u\,V.$$

Let us use the notation

$$u^{(s)}(\vec{p}) = N\sqrt{E+m}\begin{pmatrix}\chi^{(s)}\\ \dfrac{\vec{\sigma}.\vec{p}}{E+m}\chi^{(s)}\end{pmatrix} \quad , \qquad \text{for } E > 0$$

then

$$u^\dagger\,u \;= |N|^2\;\;2E$$

and the normalization choice of 2E particles in volume V requires

$$N = \frac{1}{\sqrt{V}}\;\;.$$

(This makes our discussion similar for fermions and bosons).

Number of final states

For a single particle, quantum theory restricts the number of final states in volume V with momentum in element d^3p to be

$$\frac{V\,d^3p}{(2\pi)^3}\;\;.$$

We have 2E particles in volume V, so

Number of final states per particle $= \dfrac{V\, d^3p}{(2\pi)^3\, 2E}$.

For the final particles C and D, the number of final states is

$$= \frac{V\, d^3p_C}{(2\pi)^3 2E_C} \frac{V\, d^3p_D}{(2\pi)^3 2E_D} \ .$$

Initial Flux

Let target particle B is at rest and the incident particle A is moving with velocity $\mathbf{v}_A$.

Initial flux (per unit time) = (number of beam particles passing through a unit area of target per unit time) × (number of target particles per unit volume)

$$= \frac{(S\, v_A)\,(\frac{2E_A}{V})}{S} \times (\frac{2E_B}{V}) = v_A \frac{2E_A}{V}\frac{2E_B}{V} \ .$$

Transition rate per unit time

$$W_{fi} = \frac{|T_{fi}|^2}{TV}$$

where

$$T_{fi} = (N_A\ N_B\ N_C\ N_D)\,(2\pi)^4\ \delta^4(p_D + p_C - p_A - p_B)\,(-i\,M).$$

Now

$$(2\pi)^4\ \delta^4(p_D + p_C - p_A - p_B) = (2\pi)\delta(E_C + E_D - E_A - E_B) \times (2\pi)^3 \delta^3(\vec{p}_C + \vec{p}_D - \vec{p}_A - \vec{p}_B)$$

$$= \ \lim_{T\to\infty, V\to\infty} \int_{-T/2}^{T/2} dt\, \exp\{i(E_C + E_D - E_A - E_B)\} \ \int_V d^3x\ \exp\{i(\vec{p}_C + \vec{p}_D - \vec{p}_A - \vec{p}_B)\}$$

$$= \lim_{T\to\infty, V\to\infty}(TV)\ ,$$

so that we write

$$W_{fi} = \frac{|T_{fi}|^2}{TV}$$

$$= \lim_{T\to\infty, V\to\infty} \frac{1}{V^4} \times \frac{TV|M|^2 (2\pi)^4 \delta^4 (p_C + p_D - p_A - p_B)}{TV}$$

$$\text{or} \quad W_{fi} = \lim_{V\to\infty} \frac{(2\pi)^4 \delta^4 (p_C + p_D - p_A - p_B)\ |M|^2}{V^4}.$$

Substituting expressions of transition rate per unit volume, number of final states and initial flux, we get the following expression for the differential cross section (the arbitrary volume V cancels)

$$d\sigma = \frac{|M|^2 (2\pi)^4 \delta^4 (p_C + p_D - p_A - p_B)}{v_A \times 2E_A \times 2E_B} \times \frac{d^3 p_C}{(2\pi)^3 2E_C} \times \frac{d^3 p_D}{(2\pi)^3 2E_D} .$$

Since the arbitrary volume V cancels, we may take, from now on, V =1 and so N=1. That is, we use covariant normalization of 2E particles per unit volume. Thus

$$d\sigma = \frac{|M|^2}{F}\ dQ ,$$

where the Lorentz invariant phase space factor is

$$dQ = (2\pi)^4 \delta^4 (p_C + p_D - p_A - p_B) \times \frac{d^3 p_C}{(2\pi)^3 2E_C} \times \frac{d^3 p_D}{(2\pi)^3 2E_D}$$

and the initial flux factor is

$F = v_A\ \ 2E_A\ \ 2E_B$.

Comments:

(i) The choice of N =1 means that from now on we may use the following solutions,

$\phi(x) = e^{-i p . x}$ for aboson , and $u^{(s)}(\vec{p}) = \sqrt{E+m}\begin{pmatrix} \chi^{(s)} \\ \dfrac{\vec{\sigma}.\vec{p}}{E+m}\chi^{(s)} \end{pmatrix}$, for a fermion.

(ii) For a general collinear collision between A and B, the initial flux factor is

$F = 4\sqrt{(p_A.p_B)^2 - m_A^2 m_B^2}$.

Hint: In lab frame $\mathbf{v}_A = \mathbf{p}_A / E_A$ and $\mathbf{v}_B = 0$. In a general collinear collision

$$F = |\vec{v}_A - \vec{v}_B| 2E_A 2E_B = \left|\frac{\vec{p}_A}{E_A} - \frac{\vec{p}_B}{E_B}\right| 2E_A 2E_B$$

$$= 4|\vec{p}_A E_B - \vec{p}_B E_A|$$

For a collinear collision, in CM frame $\mathbf{p}_A = -\,\mathbf{p}_B$,

$$\therefore F = 4\,(\,|\,\mathbf{p}_A|\,E_B + |\,\mathbf{p}_B|\,E_A\,)$$

$$= 4\sqrt{(p_A.p_B)^2 - m_A^2 m_B^2}\ .$$