Problem 3

An electron (charge –e, spin ½) is interacting with an EM field $A^{\mu}(x)$. Identify the interaction potential V(x) appearing in the expression for transition amplitude T_{fi} .

Show that $T_{fi} = -i\int d^4x \; j^{fi}_{\mu}(x) \; A^{\mu}(x)$ **where** $j^{fi}_{\mu}(x) = -e\overline{\psi}_f(x) \; \gamma_{\mu} \; \psi_i(x)$.

The free particle Dirac equation is

$$H\,\psi = (\boldsymbol{\alpha}\,.\,\mathbf{p} + \beta \; m)\,\psi$$

or $$(i\,\beta\frac{\partial}{\partial t} + i\,\beta\,\vec{\alpha}.\vec{\nabla} \; - m)\,\psi = 0\,.$$

Define $\gamma^{\mu} = (\beta, \beta\vec{\alpha})$, so that the free particle Dirac equation may be written as

$$(\gamma^{\mu}\hat{p}_{\mu} - m)\psi = 0\,,$$

where $\hat{p}_{\mu} = i\partial_{\mu}$. For the electron in the presence of an EM field $A^{\mu}(x)$, the minimal substitution rule suggests,

$p^{\mu} \rightarrow p^{\mu} + e\,A^{\mu}$ (classical ED)

$\hat{p}_{\mu} \rightarrow \hat{p}_{\mu} + e\hat{A}_{\mu}$ (quantum version)

Thus one may write Dirac equation for an electron in the presence of EM field as

$$(\gamma_{\mu}\hat{p}_{\mu} - m)\psi = -e\gamma_{\mu}A^{\mu}\psi$$

This is equivalent to

$$i\frac{\partial}{\partial t}\psi = (-i\vec{\alpha}.\vec{\nabla} + \beta m)\psi - e\beta\gamma_{\mu}A^{\mu}\psi$$

The first term on RHS is unperturbed Hamiltonian and the second term on RHS is the perturbation arising due to interaction of the electron with EM field. Therefore for an e^- interacting with an EM field $A^{\mu}(x)$, the interaction potential is

$V(x) = - e\, \beta\, \gamma_\mu\, A^\mu(x)$.

The transition amplitude is

$$T_{fi} = -i \int d^4x\, \psi_f^+(x)\, V(x)\, \psi_i(x)$$

or $$T_{fi} = -i \int d^4x (-e \overline{\psi}_f(x) \gamma_\mu \psi_i(x)) A^\mu(x)$$

For a spin half particle, since the charge current density is $j_\mu^{fi}(x) = -e\overline{\psi}_f(x)\gamma_\mu\psi_i(x)$, the transition amplitude can be written as

$$T_{fi} = -i \int d^4x\ \ j_\mu^{fi}(x)\ \ A^\mu(x)$$

Comments:

(i) Use of $\psi(x) = u(\mathbf{p})\, e^{-i\, p\, .\, x}$ gives

$$j_\mu^{fi}(x) = -e\overline{u}_f(\vec{p}_f)\gamma_\mu u_i(\vec{p}_i) e^{i(p_f - p_i).x}$$

(ii) Use of $A^\mu(x) = \varepsilon^\mu(\mathbf{k})\ \ e^{-i\, k\, .\, x}$ gives

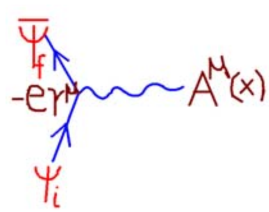
$$T_{fi} = (2\pi)^4 \delta^4(p_f - p_i - k) M$$

where the $\delta^4(\ldots.)$ represents conservation of energy and momentum, and M is

$$M = \overline{u}_f(\vec{p}_f)(ie\gamma_\mu)u_i(\vec{p}_i)\varepsilon^\mu(\vec{k})\ .$$

It is called the invariant amplitude.

(iii) Feynman diagram in configuration space represents T_{fi}



(iv) Feynman diagram in momentum space represents M