Problem 2

Obtain expression for the amplitude of transition from a state $\phi_i(x)$ to a state $\phi_f(x)$ under the action of a perturbation V(x).

Consider a free particle described by Schrodinger equation

$H_0\,\phi_n = E_n\,\phi_n$,

with

$$\int_V \phi_m^* \phi_n d^3x = \delta_{mn}\,.$$

Here H_0, the unperturbed Hamiltonian is time independent. The normalization is – one particle in a box of volume V. The particle is subject to a perturbation $V(x) \equiv V(\mathbf{x}, t)$ and is described by the equation

$$(H_0 + V(\vec{x}, t))\psi = i\frac{\partial \psi}{\partial t}\ .$$

Any solution of the time dependent Schrodinger equation can be written in the form

$$\psi(\vec{x}, t) = \sum_n a_n(t)\phi_n(\vec{x})\ e^{-iE_n t}$$

where $a_n(t)$ is unknown. It is a measure of amplitude for transition from state ϕ_i to ϕ_n . By use of the form of ψ, multiplication by ϕ_f^* , integration over volume V and use of orthogonality relation, one obtains

$$\frac{da_f}{dt} = -i\sum_n a_n(t)\int \phi_f^* V \phi_n d^3x\ e^{i(E_f - E_n)t}\ .$$

Let the initial state of the system is i (i.e., before V acts) at time t = - T/2, so $a_i(-T/2) = 1$ and $a_n(-T/2) = 0$ for $n \neq i$. Assume V to be small and transient, then

$$a_f(t') = -i\int_{-T/2}^{t'} dt \int \phi_f^* V \phi_i d^3x \quad e^{i(E_f - E_i)t}$$

At time t' = + T/2, let the perturbation cease. Then

$$a_f(T/2) = -i\int_{-T/2}^{T/2} dt \int d^3x \; \left[\phi_f(\vec{x}) \; e^{-E_f t}\right]^* \; V(\vec{x}, t) \; \left[\phi_i(\vec{x}) \; e^{-iE_i t}\right]$$

This is the expression for the transition amplitude for transition from an initial state i at time t = - T/2 to a final state f at time t = + T/2 under the action of a perturbation V(**x**,t). In covariant form, it is written as ($T_{fi} \equiv a_f(T/2)$)

$$T_{fi} = -i\int d^4x \;\; \Phi_f^*(x) \; V(x) \; \Phi_i(x)$$