Problem 15

State and verify (I) the contraction properties and (II) trace theorems for the γ-matrices.

➢ (I) The α^i and β matrices appearing in Dirac equation are

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

The γ-matrices are defined as

$$\gamma^0 = \beta \ , \qquad \gamma^k = \beta\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \quad \text{and} \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 .$$

These have the following properties:

$$(\gamma^0)^2 = 1, \ \ (\gamma^k)^2 = -1, \quad (\gamma^5)^2 = 1, \ \ (\gamma^5)^\dagger = \gamma^5 \ , \ \ \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} ;$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\, g^{\mu\nu}, \qquad \gamma^\mu \gamma^5 + \gamma^5 \gamma^\mu = 0 \ .$$

The contraction properties are :

(i) $\gamma_\mu \gamma^\mu = 4$.

Proof: $\gamma_\mu \gamma^\mu = \gamma_0 \gamma^0 + \gamma_i \gamma^i = (\gamma^0)^2 - (\gamma^i)^2 = 1 - (-1-1-1) = 4$.

(ii) $\gamma_\mu \gamma^\alpha \gamma^\mu = -2\gamma^\alpha$.

Proof: $\gamma_\mu \gamma^\alpha \gamma^\mu = \gamma_\mu (2 g^{\alpha\mu} - \gamma^\mu \gamma^\alpha) = 2\gamma^\alpha - \gamma_\mu \gamma^\mu \gamma^\alpha = 2\gamma^\alpha - 4\gamma^\alpha = -2\gamma^\alpha$.

(iii) $\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu = 4 g^{\alpha\beta}$.

Proof: $\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu = \gamma_\mu \gamma^\alpha (2 g^{\beta\mu} - \gamma^\mu \gamma^\beta) = 2\gamma^\beta \gamma^\alpha - \gamma_\mu \gamma^\alpha \gamma^\mu \gamma^\beta = 2\gamma^\beta\gamma^\alpha + 2\gamma^\alpha\gamma^\beta = 4 g^{\alpha\beta}$.

(iv) $\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\mu = -2\gamma^\gamma \gamma^\beta \gamma^\alpha$

Proof: Try yourself.

(v) $\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \gamma^\mu = 2(\gamma^\delta \gamma^\alpha \gamma^\beta \gamma^\gamma + \gamma^\gamma \gamma^\beta \gamma^\alpha \gamma^\delta)$.

Proof: Try yourself.

Comments: Using the notation $\not{a} = a_\beta \gamma^\beta$ these contraction properties may be written as (see Bjorken and Drell vol.1 page 105):

$\gamma_\mu \gamma^\mu = 4$,

$\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$,

$\gamma_\mu \not{a}\not{b} \gamma^\mu = 4a.b$,

$\gamma_\mu \not{a}\not{b}\not{c} \gamma^\mu = -2\not{c}\not{b}\not{a}$,

$\gamma_\mu \not{a}\not{b}\not{c}\not{d} \gamma^\mu = 2[\not{d}\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}\not{d}]$.

Some useful properties of $\varepsilon^{\alpha\beta\gamma\delta}$:

$\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta} = -24$,

$\varepsilon^{\alpha\beta\gamma\mu}\varepsilon_{\alpha\beta\gamma\nu} = -6 g^\mu_\nu$,

$\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\sigma\rho} = -2(g^\mu_\sigma g^\nu_\rho - g^\mu_\rho g^\nu_\sigma)$.

(II) Trace theorems:

Tr 1 = 4

Trace of an odd number of γ_μ's vanishes.

$\mathrm{Tr}\,(\not{a}\,\not{b}) = 4\, a\,.\,b$

$\mathrm{Tr}\,(\not{a}\,\not{b}\,\not{c}\,\not{d}) = 4\,[\,a\,.\,b\;\; c\,.\,d\; - \;a\,.\,c\;\; b\,.\,d\; + \;a\,.\,d\;\; b\,.\,c\,]$

$\mathrm{Tr}\;\; \gamma_5 = 0$

$\mathrm{Tr}\, \gamma_5\; \not{a}\; \not{b}\; = 0$

$\mathrm{Tr}\,(\gamma_5\; \not{a}\; \not{b}\; \not{c}\; \not{d}\,) = 4\, i\, \varepsilon_{\mu\nu\lambda\sigma}\;\; a^\mu b^\nu c^\lambda d^\sigma$,

where $\varepsilon_{\mu\nu\lambda\sigma}$ = + 1(− 1) for μ, ν, λ, σ an even (odd) permutation of 0, 1, 2, 3; and $\varepsilon_{\mu\nu\lambda\sigma} = 0$ if any two indices are the same.

Proof:

Note that Tr(AB) = Tr(BA), therefore

$\mathrm{Tr}(\not{a}\;\not{b}\,) = ½\;\; \mathrm{Tr}(\not{a}\;\not{b} + \not{b}\;\not{a}\,) = a\,.\,b\; \mathrm{Tr}(1) = 4\; a\,.\,b$

Alternative:

$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = \mathrm{Tr}(2g^{\mu\nu} - \gamma^\nu\gamma^\mu) = 8\,g^{\mu\nu} - \mathrm{Tr}(\gamma^\nu\gamma^\mu)$

$\qquad = 8\,g^{\mu\nu} - \mathrm{Tr}\,\gamma^\mu\gamma^\nu$ (because $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$)

$\therefore\ \mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4\,g^{\mu\nu}$.

Alternative:

$\mathrm{Tr}\,\not a\,\not b = \mathrm{Tr}(\tfrac14\,\not a\,\not b\,\gamma_\mu\gamma^\mu) = \tfrac14\,\mathrm{Tr}(\gamma^\mu\,\not a\,\not b\,\gamma_\mu) = \tfrac14\,\mathrm{Tr}(4\,a\,.\,b) = a\,.\,b\,\mathrm{Tr}(1) = 4\,a\,.\,b$

α^k and β are traceless matrices, therefore $\mathrm{Tr}(\gamma^\mu) = 0$.

$\mathrm{Tr}(\gamma^\alpha\,\gamma^\beta\,\gamma^\gamma) = \mathrm{Tr}((\gamma^5)^2\,\gamma^\alpha\,\gamma^\beta\,\gamma^\gamma) = \mathrm{Tr}(\gamma^5\,\gamma^\alpha\,\gamma^\beta\,\gamma^\gamma\,\gamma^5) = (-)\,\mathrm{Tr}(\gamma^5\,\gamma^\alpha\,\gamma^\beta\,\gamma^5\,\gamma^\gamma)$

$\qquad = (-)^3\,\mathrm{Tr}(\gamma^5\,\gamma^5\,\gamma^\alpha\,\gamma^\beta\,\gamma^\gamma) = -\,\mathrm{Tr}((\gamma^5)^2\,\gamma^\alpha\,\gamma^\beta\,\gamma^\gamma) = -\,\mathrm{Tr}(\gamma^\alpha\,\gamma^\beta\,\gamma^\gamma)$,

$\therefore\ \mathrm{Tr}(\gamma^\alpha\,\gamma^\beta\,\gamma^\gamma) = 0$.

(Comment: generalize it, see Bjorken and Drell)

$\mathrm{Tr}(\gamma_5\,\not a\,\not b) = \tfrac14\,\mathrm{Tr}(\gamma_5\,\not a\,\not b\,\gamma_\mu\gamma^\mu) = \tfrac14\,\mathrm{Tr}(\gamma^\mu\,\gamma_5\,\not a\,\not b\,\gamma_\mu) = -\,\tfrac14\,\mathrm{Tr}(\gamma_5\,\gamma^\mu\,\not a\,\not b\,\gamma_\mu)$

$\qquad = -\,\tfrac14\,\mathrm{Tr}(\gamma_5\,4\,a\,.\,b) = -\,a\,.\,b\,\mathrm{Tr}(\gamma_5) = 0.$

$\mathrm{Tr}(\gamma^\mu\,\gamma^\nu\,\gamma^\sigma\,\gamma^\rho) = \mathrm{Tr}(\gamma^\mu\,\gamma^\nu\,(2\,g^{\sigma\rho} - \gamma^\rho\,\gamma^\sigma)) = 2\,g^{\sigma\rho}\,\mathrm{Tr}(\gamma^\mu\,\gamma^\nu) - \mathrm{Tr}(\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma)$

$\qquad = 8\,g^{\sigma\rho}\,g^{\mu\nu} - \mathrm{Tr}(\gamma^\mu\,(2\,g^{\nu\rho} - \gamma^\rho\,\gamma^\nu)\,\gamma^\sigma)$

$\qquad = 8\,g^{\mu\nu}\,g^{\sigma\rho} - 8\,g^{\mu\sigma}\,g^{\nu\rho} + \mathrm{Tr}(\gamma^\mu\,\gamma^\rho\,\gamma^\nu\,\gamma^\sigma)$

$\qquad = 8\,g^{\mu\nu}\,g^{\sigma\rho} - 8\,g^{\mu\sigma}\,g^{\nu\rho} + \mathrm{Tr}((2\,g^{\mu\rho} - \gamma^\rho\,\gamma^\mu)\,\gamma^\nu\,\gamma^\sigma)$

$\qquad = 8\,g^{\mu\nu}\,g^{\sigma\rho} - 8\,g^{\mu\sigma}\,g^{\nu\rho} + 8\,g^{\mu\rho}\,g^{\nu\sigma} - \mathrm{Tr}(\gamma^\rho\,\gamma^\mu\,\gamma^\nu\,\gamma^\sigma)$

$\qquad = 8\,g^{\mu\nu}\,g^{\sigma\rho} - 8\,g^{\mu\sigma}\,g^{\nu\rho} + 8\,g^{\mu\rho}\,g^{\nu\sigma} - \mathrm{Tr}(\gamma^\mu\,\gamma^\nu\,\gamma^\sigma\,\gamma^\rho)$

$\therefore\quad \mathrm{Tr}(\gamma^\mu\,\gamma^\nu\,\gamma^\sigma\,\gamma^\rho) = 4\,[\,g^{\mu\nu}\,g^{\sigma\rho} - g^{\mu\sigma}\,g^{\nu\rho} + g^{\mu\rho}\,g^{\nu\sigma}\,].$

(Comment: the above implies

$$\mathrm{Tr}\,(\not a\,\not b\,\not c\,\not d\,) = 4\,[\,a\,.\,b\;\;c\,.\,d\; - \;a\,.\,c\;\;b\,.\,d\; + \;a\,.\,d\;\;b\,.\,c\,]$$

Show it.)

Look at the various components of

$$\mathrm{Tr}\,(\gamma_5\ \not{a}\ \not{b}\ \not{c}\ \not{d}\ )$$

For a non-vanishing contribution, all components of *a, b, c, d,* should be different and the total contribution is the sum of the various combinations of components multiplied by the sign of the permutation. To fix the sign, consider

$$\mathrm{Tr}(\ \gamma_5\ \gamma_0\ \gamma_1\ \gamma_2\ \gamma_3\ a^0\ b^1\ c^2\ d^3\ )\ = i\ \varepsilon_{0123}\ \ a^0\ b^1\ c^2\ d^3\ \mathrm{Tr}((\gamma^5)^2) \qquad \text{(think)}$$

$$= 4\ i\ \varepsilon_{0123}\ \ a^0\ b^1\ c^2\ d^3\ .$$

This leads to

$$\mathrm{Tr}(\gamma_5\ \not{a}\ \not{b}\ \not{c}\ \not{d}\ ) = 4\ i\ \varepsilon_{\mu\,\nu\,\lambda\,\sigma}\ \ a^\mu\, b^\nu\, c^\lambda\, d^\sigma\ ,$$

(For more theorems, see Bjorken and Drell, pages 104 – 105.)