Problem 11

Define Mandelstam variables, s, t, u, for the scattering process A + B → C + D. Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$.

- Consider the process $A(p_A) + B(p_B) \to C(p_C) + D(p_D)$. Two particles A and B of (four) momenta p_A, p_B and masses m_A, m_B scatter to particles C and D of momenta p_C, p_D and masses m_C, m_D. The Mandelstam variables are defined by

$$s = (p_A + p_B)^2 = (p_C + p_D)^2$$
$$= m_A^2 + m_B^2 + 2E_A E_B - 2\vec{p}_A.\vec{p}_B,$$
$$t = (p_A - p_C)^2 = (p_B - p_D)^2$$
$$= m_A^2 + m_C^2 - 2E_A E_C + 2\vec{p}_A.\vec{p}_C,$$
$$u = (p_A - p_D)^2 = (p_B - p_C)^2$$
$$= m_A^2 + m_D^2 - 2E_A E_D + 2\vec{p}_A.\vec{p}_D.$$

Adding these variables we find that

$$s + t + u = 3m_A^2 + m_B^2 + m_C^2 + m_D^2 + 2E_A(E_B - E_C - E_D) - 2\vec{p}_A.(\vec{p}_B - \vec{p}_C - \vec{p}_D).$$

Conservation of energy and three momentum requires

$E_B - E_C - E_D = -E_A$ and $\mathbf{p}_B - \mathbf{p}_C - \mathbf{p}_D = -\mathbf{p}_A$, therefore

$$s + t + u = 3m_A^2 + m_B^2 + m_C^2 + m_D^2 - (2E_A^2 - 2\vec{p}_A^2),$$

thus the Mandelstam variables satisfy

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2.$$

Comments:

(i) s, t, u are Lorentz-invariant variables. The invariant amplitude M is often expressible as functions of these invariant variables.

(ii) For A B → C D, s is the square of the total CM energy of the process. The process is called s-channel process (where $\sqrt{s}$ = total CM energy).

(iii) When A B → C D is the s-channel process, then $A\bar{D} \to C\bar{B}$ is called the u-channel process and $B\bar{D} \to C\bar{A}$ is called the t-channel process.

(See Fig. 4.6 of the book (Quarks and Leptons by Halzen and Martin, John Wiley & Sons © 1984) and related discussion).