Determination of coefficient of rigidity as a function of temperature using torsion oscillator (resonance method)

13.

Detailed explanation of the principles involved in this experiment:

1. Torsion Oscillator – a simple way of measuring modulus of rigidity.

A torsion oscillator is shown in Fig.1. It is a clamp (torsion head) in which one end of a cylindrical wire, W is rigidly clamped. A cylindrical rod (bar) is attached to the other end of the wire.

Fig.1 Torsion Oscillator

The wire consists the axis of the disc.

When the disc is given a twist of angle θ, by applying a pair of tangential forces F_1 and F_2, by applying a torque $\tau = F_1 d$, then a restoring torque, τ , develops in the wire. It is given by (the small twist angle)

$$\tau = -C\theta \qquad (1.1)$$

where C is the restoring torque per unit angle of twist

For a cylindrical wire of radius r, length l and modulus of rigidity η , it can be proved theoretically that

$$C = \frac{\pi \eta r^4}{2l} \qquad (1.2)$$

Now, if the external pair of tangential forces is withdrawn, the disc performs angular oscillations under the action of the restoring torque.

If I is the moment of inertial of the disc about its axis, then the equation of angular motion (Eq.(1.1)) is

$$I\frac{d^2\theta}{dt^2} = -C\theta \qquad (1.3)$$

This is equation of angular simple harmonic oscillations. The time period, T, and the angular frequency, ω_0 , of the oscillations are respectively

$$T = 2\pi\sqrt{\frac{I}{C}} \qquad (1.4)$$

$$\omega_0 = \sqrt{\frac{C}{I}} \qquad (1.5)$$

Using Eq.(1.2), 4 and 5), we can write

$$\eta = \frac{8\pi I l}{r^4 T^2} \qquad (1.6)$$

or

$$\eta = \frac{2\omega_0^2 l I}{\pi r^4} \qquad (1.7)$$

The moment of inertia of a circular disc about its axis is:

$$I = \frac{1}{2}Mr^2 \qquad (1.8)$$

By measuring the time period (angular frequency) of oscillations, length of the wire, radius of the wire and moment of inertia of the disc, the modulus of rigidity of the material of the wire is determined by using Eqs. (1.6 or 1.7).

In the above discussion, we have ignored damping. When damping is included, Eq.(1.3) gets modified.

2.Description of torsion oscillator developed at CIPE (University of Rajasthan):

This experiment is based on an apparatus developed at CIPE (University of Rajasthan, Jaipur). See also "Temperature variation of modulus of rigidity and internal friction: An experiment with torsional oscillator", Gauri Shankar, V. K. Gupta, B.Sand and N. K. Sharma, Am. J. Phys. 53, 1192 (1985) for original figures and discussion.

The material whose modulus of rigidity is to be studied, is taken in the form of a specimen rod of length l ~ 0.1 m and radius r ~ 1.1 mm. It is suspended vertically by clamping its upper end in a rigid support (I in Fig.2). To the free end of the rod is attached an auxiliary body, an aluminum bar (J in Fig.2), which carries a small permanent magnet (marked as N and S in Fig.2). This apparatus is kept surrounded by a heater unit to maintain it at a desired temperature (2 in Fig.2).

The purpose of the auxiliary body is two fold, i.e., to control the resonant frequency of the system and to isolate the coupling magnet from the heater

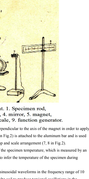

Fig. 2 Experimental setup (torsion oscillator). 1. Specimen rod, 2. Heater unit, 3. auxiliary body, 4. mirror, 5. magnet, 6. Helmholtz coil, 7. lamp, 8. scale, 9. function generator.

A Helmholtz coil (6 in Fig.2) is used with its axis perpendicular to the axis of the magnet in order to apply an external torque on the system. A small concave mirror (4 in Fig.2) is attached to the aluminum bar and is used for measurements of angular oscillations, using a lamp and scale arrangement (7, 8 in Fig.2).

Resistance of the heater coil is calibrated in terms of the specimen temperature, which is measured by an iron-constantan thermocouple. The calibration curve is used to infer the temperature of the specimen during experiments.

A function generator (9 in Fig.2) is used to generate sinusoidal waveforms in the frequency range of 10 Hz – 1 kHz in two decades steps. The signal is fed to Helmholtz coil to produce torsional oscillations in the system.

3.Theory of forced oscillations:

Let a specimen rod and an auxiliary body attached to it be set into oscillations under the influence of an external oscillatory torque $\tau = \tau_0 e^{i\omega t}$. Let the moment of inertia of the system (specimen rod plus auxiliary body) is I, the angular oscillation is θ , and the torsional constant of the system is C.

In this experiment, the moment of inertia of the specimen rod is less than 1% that of the auxiliary body. Hence the moment of inertia of the system may be taken to be equal to the moment of inertia of the auxiliary body.

The effective value of the torsion constant, C, in this experiment, is given by

$$\frac{1}{C_e} = \frac{1}{C} + \frac{1}{C_s} \qquad (3.1)$$

where C_s and C_e are the torsion constants for the specimen rod and the auxiliary body, respectively. If $C_s \gg C_e$, then $C \approx C_e$. In this experiment $C_s / C > 200$, therefore the assumption $C \approx C_e$ is justified.

The equation of motion of this system may be written as

$$I\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + C\theta = \tau_0 e^{i\omega t} \qquad (3.2)$$

Since the restoring torque is mainly provided by the specimen rod, the torsion constant may be expressed as

$$C = \frac{\pi \eta r^4}{2l} \qquad (3.3)$$

where η , r and l are, respectively, the modulus of rigidity, radius and length of the specimen rod.

Steady state solution of Eq.(3.2), after the transients are over, is given by

$$\theta = \theta_0 e^{i(\omega t - \delta)} \qquad (3.4)$$

Use of Eq.(3.4) in Eq.(3.2) gives

$$\theta_0 = \frac{\tau_0/C}{\left[\left(1 - \frac{I\omega^2}{C}\right)^2 + \frac{\gamma^2\omega^2}{C^2}\right]^{1/2}} \qquad (3.5)$$

For the case of low damping, θ_0 will be maximum for

$$\omega = \sqrt{\frac{C}{I}} = \omega_0 \quad (3.6)$$

Substituting for C from Eq.(3) and solving for η , we obtain

$$\eta = \frac{2\omega_0^2 l I}{\pi r^4} \qquad (3.7)$$

A knowledge of I, l and ω_0 determines η.

The torsion constant C depends on temperature through η. As the temperature rises, because of more lattice vibrations, the shear rigidity of the specimen decreases, i.e., it softens and as a result C decreases. This in turn decreases the resonance angular frequency ω_0 .

The full width of the resonance curve at the half maximum amplitude is determined using Eq.(3.5). It is given by

$$\Delta\omega = \frac{\gamma}{I\omega_0} \qquad (3.8)$$

It is a measure of internal friction. When is the temperature, less is the value of C (and η) and more is the value of $\Delta\omega$. The reverse of quality factor Q is generally regarded as a measure of internal friction. It is given by

$$Q = \frac{I\omega_0}{\gamma} = \frac{\omega_0}{\Delta\omega} \qquad (3.9)$$

With increase in temperature, the internal friction increase. As a result the width of the resonance curve increases.

4. Temperature dependence of modulus of rigidity:

The temperature dependence of the modulus of rigidity for some materials is shown in Fig.3. The modulus of rigidity appears to decrease linearly with increase in temperature. Quantitatively, the temperature dependence of η is given by

$$\eta_T = \eta_0 - \eta T \qquad (4.1)$$

where η_0 = modulus of rigidity at temperature T and η_T , η and T are free parameters, usually determined from the experimental data.

Fig.3 Variation of rigidity with temperature

Approximating

$$e^{\alpha T} = 1 + \frac{\alpha T}{1!} + \ldots \quad (\text{for } 0 < T), \quad (4.2)$$

Using eq (4.2)(except T^2 and higher order terms) in Eq.(4.1), we find

$$\eta_T = \eta_0 - \left(\frac{\eta}{\alpha}\right)^{1/2} \qquad (4.2)$$

Thus a graph between η_T and temperature T is a straight line with negative slope, as seen in Fig.4.

4. Methodology adopted in measure η

(i) When in current input from the function generator is fed to the Helmholtz coil, it produces a magnetic field whose magnitude and direction follows the ac signal, i.e., the magnetic field produced is also oscillatory. As a result an oscillatory torque acts on the magnet (Fig.4). This Fig.4 Sinusoidal current in the Helmholtz coil produces an oscillatory torque acts on the magnet (Fig.4). The magnet attached at one end of auxiliary rod (J) specimen rod and attached auxiliary aluminum rod under goes forced oscillations.

(ii) The amplitude of the angular oscillation performed by the specimen rod- auxiliary rod system is measured by the lamp and scale arrangement.

(iii) The amplitude of oscillations are recorded at various frequencies in the neighborhood of the resonant frequency of the system. A graph is obtained (Fig.5) from which the resonant frequency ω_0 is inferred.

(iv) The temperature of the specimen rod is determined from the current in the heating coil and calibration curve between current and temperature.

(v) The process is repeated for various temperatures of the specimen and response curve plotted (Fig.6).

(VI) For each curve, temperature and the resonance frequency is noted. As a given temperature, the modulus of rigidity of the specimen rod is determined by using the relation:-

$$\eta = \frac{2\omega_0^2 l I}{\pi r^4}$$

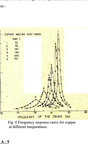

Fig.5 Amplitude resonance curve

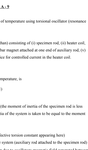

Fig. 6 Frequency response curve for copper at different temperature

Experiment.A.- 3

Object:

To determine the coefficient of rigidity as a function of temperature using torsional oscillator (resonance method).

Apparatus:

Apparatus developed by CPOE (University of Rajasthan) consisting of (1) specimen rod, (2) heater coil, (in) auxiliary bar(J) with a small mirror, (iv) a small bar magnet attached at one end of auxiliary rod (v) lamp and scale arrangement, (vi) function generator, device for controlled current in the heater coil.

Theory:

The equation of motion for the system (Fig.3), at any fixed temperature, is

$$I\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + C\theta = \tau_0 e^{i\omega t} \qquad (1)$$

where

I = moment of inertia of the auxiliary aluminum rod (the moment of inertia of the specimen rod is less than 1% that of the auxiliary body; hence the moment of inertia of the system is taken to be equal to the moment of inertia of the auxiliary body).

γ = the damping coefficient.

C = torsion constant of the specimen rod (it is the effective torsion constant appearing here)

θ = instantaneous angular displacement (twist) of the system (auxiliary rod attached to the specimen rod)

$\tau_0 e^{i\omega t}$ = instantaneous torque acting on the system due to oscillatory magnetic field generated between Helmholtz coils.

The moment of Inertia of the auxiliary rod is

$$I = \frac{1}{2}Mr^2 \qquad (2)$$

where

R = radius of the rod,

L = length of the rod,

ρ = density of aluminum rod.

The torsion constant C (also known as restoring couple per unit twist) is given by

$$C = \frac{\pi \eta r^4}{2L} \qquad (3)$$

where

η = modulus of rigidity of the material of the rod,

r = radius of the specimen rod, and

l = length of the specimen rod.

Steady state solution of Eq.(1), after the transients are over, is given by

$$\theta = \theta_0 e^{i(\omega t - \delta)} \qquad (4)$$

where the amplitude of oscillations is

$$\theta_0 = \frac{\tau_0/C}{\left[\left(1 - \frac{I\omega^2}{C}\right)^2 + \left(\frac{\gamma\omega}{C}\right)^2\right]^{1/2}} \qquad (5)$$

For the case of low damping, the resonance (Fig.5) occurs at angular frequency ω_0 given by

$$\omega_0 = \sqrt{\frac{C}{I}} \qquad (6)$$

Using Eq.(3) and Eq.(6), we get the following formula for the modulus of rigidity :

$$\eta = \frac{2\omega_0^2 l I}{\pi r^4} \qquad (6)$$

Substitution of values of l, r and ω_0 determines η.

Qualitatively, the temperature dependence of the modulus of rigidity is expressed as

$$\eta_T = \eta_0 - \eta T$$

where η_0 = modulus of rigidity at temperature T_0 and η_0 , and η are free parameters (determined from the experimental data)

A graph between η_T and temperature T is a straight line with negative slope, as seen in Fig.III.

Figures:

Fig.I Experimental arrangement. 1. Specimen rod, 2. heater coil, 3. auxiliary body, 4. mirror, 5. magnet, 6. Helmholtz coil, 7. lamp, 8. scale, 9. function generator.

Fig.III Variation of rigidity with temperature

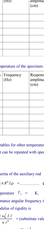

Fig. II Frequency response curve at different Temperatures.

Observations:

1. Length of aluminum rod (l)= ..... cm = ... m
2. Radius of aluminum rod (R) = ... cm = ... m
3. Density of aluminum rod (ρ)= ..... g/cm^3 = ... kg/m^3
4. Length of specimen (copper) rod (l)= ..... cm = ... m
5. Radius of specimen (copper) rod (r)= ..... cm = ... m

Table I For room temperature, the temperature of the specimen rod (T_1) = ... °C = ... K

S.No.	Frequency (Hz)	Response amplitude (cm)	Resonance frequency ω_0 (Hz) (from graph)	Modulus of rigidity η N m^{-2} (Calculated)
(1)				
(2)				
(3)				
(4)				
(5)				
(6)				
(7)				
(8)				
(9)				
(10)				

Table II For temperature of the specimen rod (T_2) = ... °C = ... K

S.No.	Frequency (Hz)	Response amplitude (cm)	Resonance frequency ω_0 (Hz) (from graph)	Modulus of rigidity η N m^{-2} (Calculated)
(1)				
(2)				
(3)				
(4)				
(5)				
(6)				
(7)				
(8)				
(9)				
(10)				

Similarly make tables for other temperatures.

(The experiment can be repeated with specimen rods of iron, silver, etc.)

Calculations:

1. Moment of inertia of the auxiliary rod

$I = \frac{1}{2}MR^2 = \frac{1}{2}\rho\pi R^4 L$ ..... $kg\,m^2$ (calculate)

2. For room temperature $T_1 = K$,

the resonance angular frequency is = $2\pi\nu_0$ = ... rad/s.

The modulus of rigidity is

$\eta = \frac{2\omega_0^2 l I}{\pi r^4}$ = (substitute values and calculate)

η_1 = ... N m^{-2}

3. Repeat Step 2 for other temperatures of the specimen rod.

4. Plot graph between η and T

Result:

1. Modulus of rigidity of the specimen (copper) at room temperature = ... N m^{-2}
2. The modulus of rigidity decreases with increase in temperature of the material, as shown in the graph for η vs T

Precautions:

1. The bar hanged at one end of auxiliary rod should be small in size and placed in the centre of Helmholtz coils.
2. Observation of response amplitude should be taken when the specimen rod temperature reaches a steady state.
3. The response amplitude near the resonance should be measured for small steps of frequency change.
4. The current in the Helmholtz coils should be kept constant during one set of observations for a given temperature.
5. The temperature of the specimen rod should not be increased beyond specified limits.